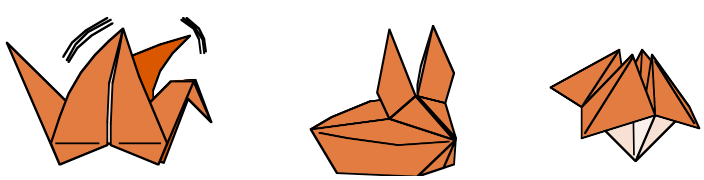
From Flapping Birds to Space Telescopes:
The Modern Science of Origami

Robert J. Lang

Background

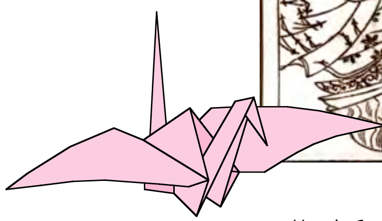
- Origami
- Traditional form



- Modern extension
- Most common version: One Sheet, No Cuts

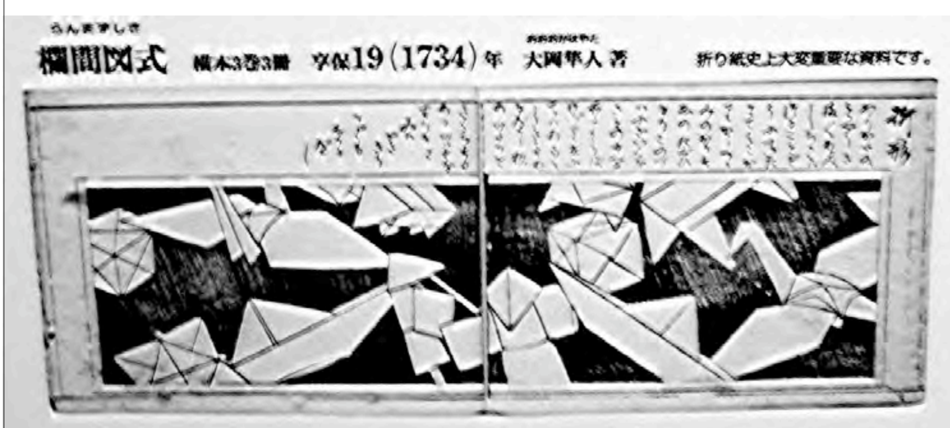
Evolution of origami

- Right: origami circa 1797.
- The traditional "tsuru" (crane)



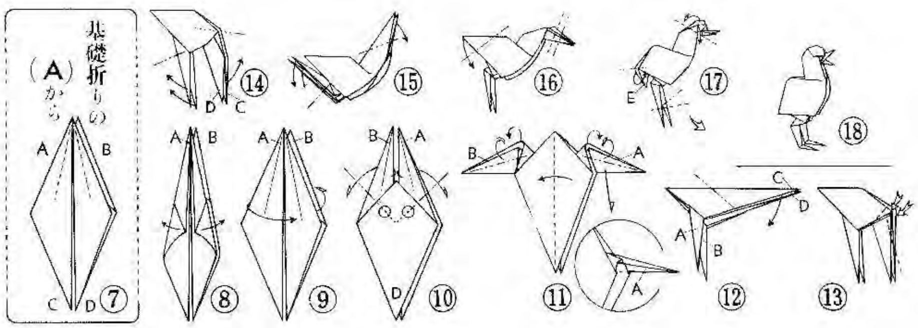
Even earlier…

- Japanese newspaper from 1734: Crane, boat, table, "yakko-san"



Modern Origami

Reborn by Yoshizawa



A. Yoshizawa, *Origami Dokuhon I*

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Origami Today

- “Black Forest Cuckoo Clock,” designed in 1987
- One sheet, no cuts
- 216 steps
  - not including repeats
- Several hours to fold


Ibex


Klein Bottle

What Changed?

- Origami was discovered by mathematicians.
- Or rather, mathematical principles
- 1950-2000…
  - From about 100…
  - …to over 36,000! (see http://www.origamidatabase.com).

The Technical Revolution

- The connection between art and science is made by mathematics.

Origami Mathematics

- The mathematics underlying origami addresses three areas:
  - Existence (what is possible)
  - Complexity (how hard it is)
  - Algorithms (how do you accomplish something)

- The scope of origami math include:
  - Plane Geometry
  - Trigonometry
  - Solid Geometry
  - Calculus and Differential Geometry
  - Linear Algebra
  - Graph Theory
  - Group Theory
  - Complexity/Computability
  - Computational Geometry

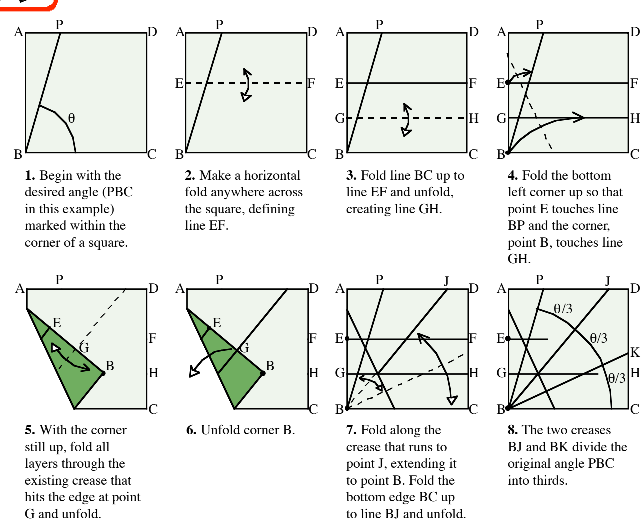
Geometric Constructions

- What shapes and distances can be constructed entirely by folding?
- Analogous to "compass-and-straightedge," but more general

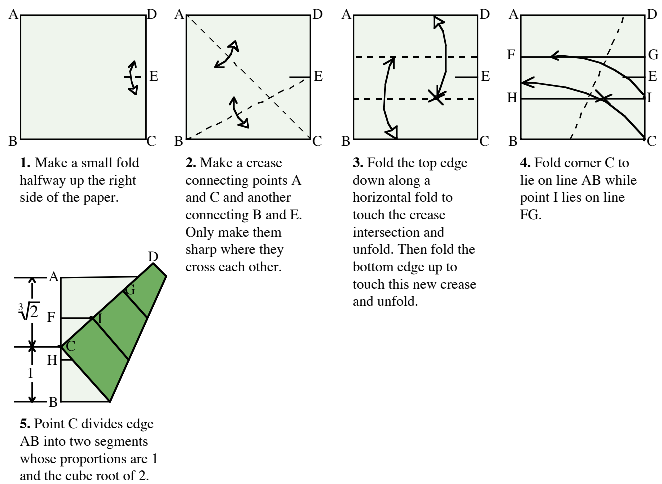
The Delian Problems

- Trisect an angle
- Double the cube
- Square the circle
- All three are impossible with compass and unmarked straightedge, but:

Hisashi Abe's Trisection



Peter Messer's Cube Doubling



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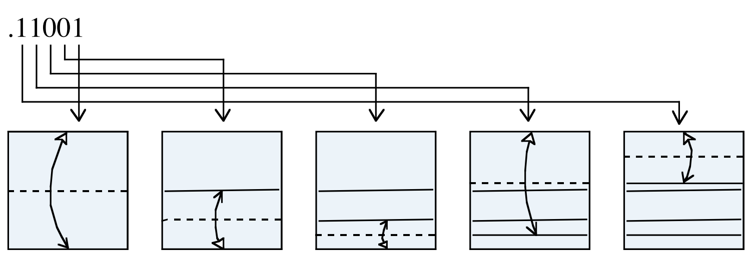
Binary Approximation for Distance

- Any distance can be approximated to $1/N$ using $\log_2 N$ folds taken from its binary expansion
- Example: 0.7813 ~ 25/32 = $.11001_2$



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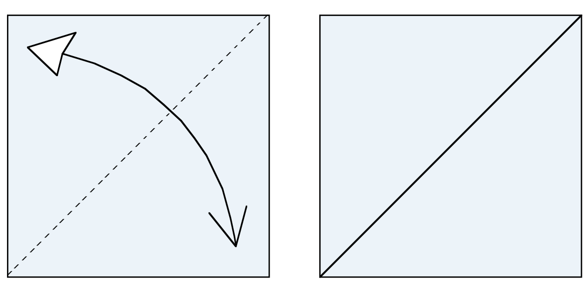
Generalize Constructions

- The binary algorithm is a special answer to a general question:

- Starting with a blank square,
- for a given point or line,
- construct an folding sequence accurate to a specified error,
- defining every fold in the sequence in terms of preexisting points and lines.

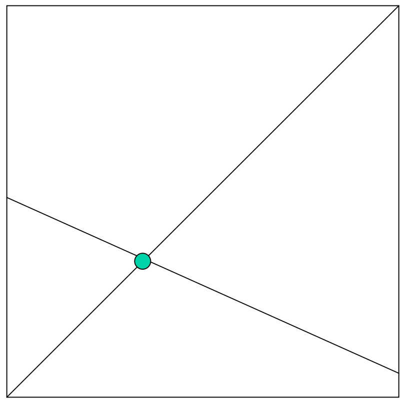
Building Blocks

- Points and Lines (creases)



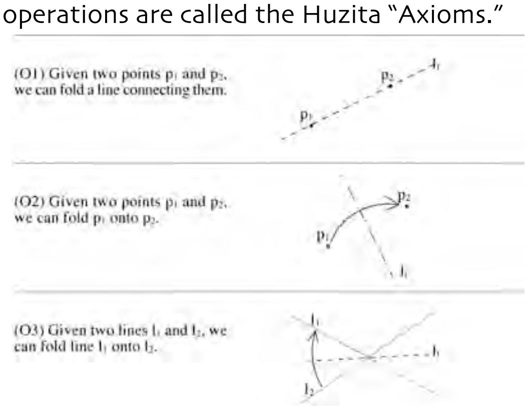
Points

A point (mark) can only be defined as the intersection of two lines. But a line (fold) can be made in many ways…



Lines

- For many years, it was thought that there were only six ways to define a fold.
- The six operations are called the Huzita "Axioms."



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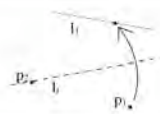
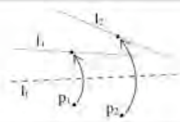
Huzita Axioms 2

(O4) Given a point p_1 and a line l_1, we can make a fold perpendicular to l_1 passing through the point p_1.

(O5) Given two points p_1 and p_2 and a line l_1, we can make a fold that places p_1 onto l_1 and passes through the point p_2.



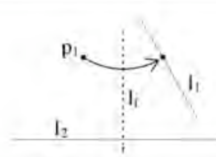
(O6) Given two points p_1 and p_2 and two lines l_1 and l_2, we can make a fold that places p_1 onto line l_1 and places p_2 onto line l_2.



Hatori's Axiom

- In 2002, Koshiro Hatori discovered a seventh "axiom."

(O7) Given a points p_1 and two lines l_1 and l_2, we can make a fold perpendicular to l_2 that places p_1 onto line l_1.



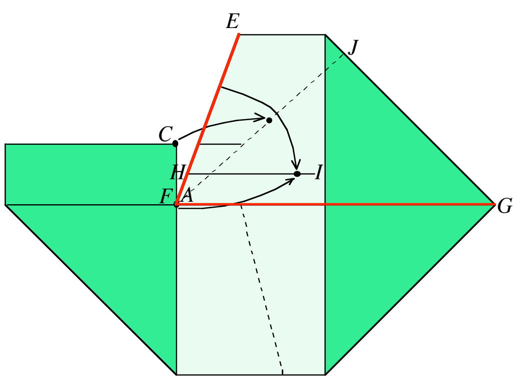
- In 2006, it was observed that Jacques Justin had identified all 7 in 1989.
- It has since been proven that these seven are the only ways to define a single fold.

Geometric Constructions

- One-fold-at-a-time origami can solve exactly:
  - All quadratic equations with rational coefficients
  - All cubic equations with rational coefficients
  - Angle trisection (Abe, Justin)
  - Doubling of the cube (Messer)
  - Regular polygons for $N=2^i3^j\{2^k3^l+1\}$ if last term is prime (Alperin, Geretschläger)
    - All regular N-gons up to N=20 except N=11

Simultaneous Creases

- If you allow forming *two* creases at one time, higher-order equations are possible.
- An angle quintisection!



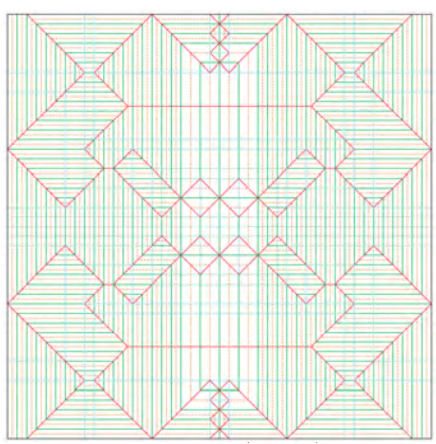
- Quintisections are impossible with only Huzita (one-fold-at-a-time) axioms.
- There are over 400 two-fold-at-a-time "axioms."

More simultaneous

- What about N-at-a-time folding?

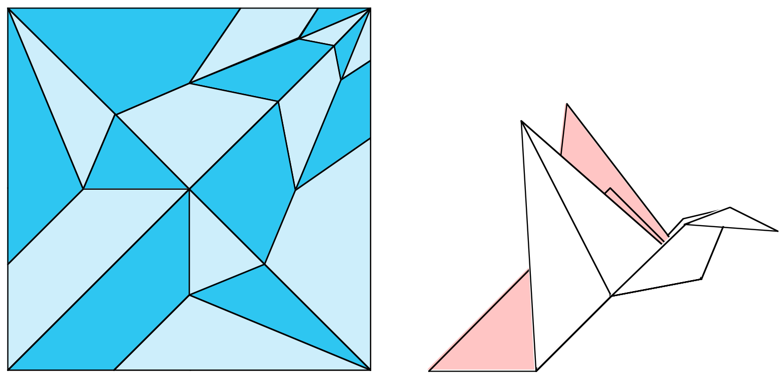
Crease Patterns

- The design of an origami figure is encoded in the crease pattern
- What constraints are there on such patterns?



Properties of Crease Patterns

- 2-colorability
- Every flat-foldable origami crease pattern can be colored so that no 2 adjacent facets are the same color with only 2 colors.



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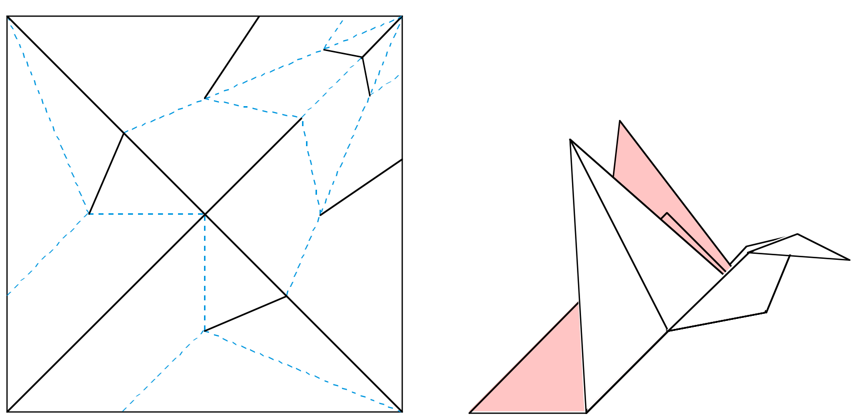
Mountain-Valley Counting

- Maekawa Condition:
  - At any interior vertex, M – V = ±2



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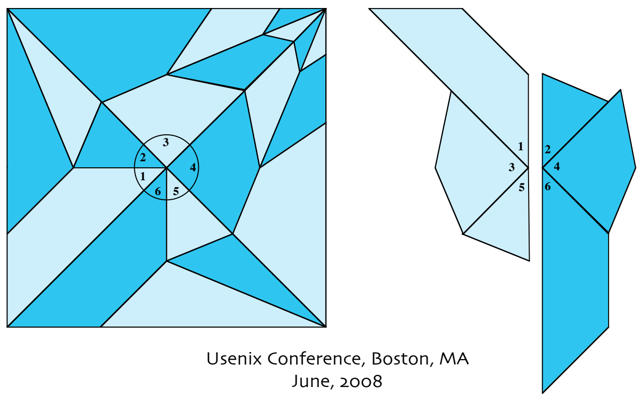
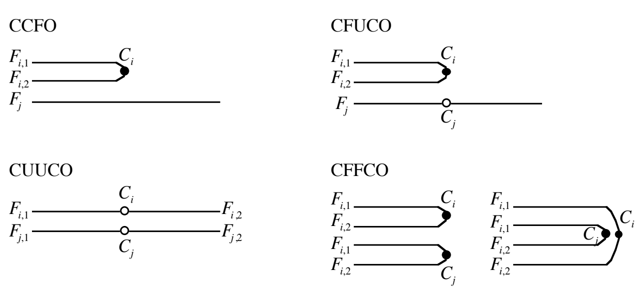
Angles Around a Vertex

- Kawasaki Condition:
  - Alternate angles around a vertex sum to a straight line
  - Independently discovered by Kawasaki, Justin, and Huffman
  - Generalized to 3D by Hull & belcastro



Layer Ordering

- A complete description of a folded form includes the layer ordering among overlapping facets (M-V is not enough!)
- Four necessary conditions were enumerated by Jacques Justin
- Pictorially, these are the “legal” layer orderings between layers, folded creases, and unfolded (flat) creases



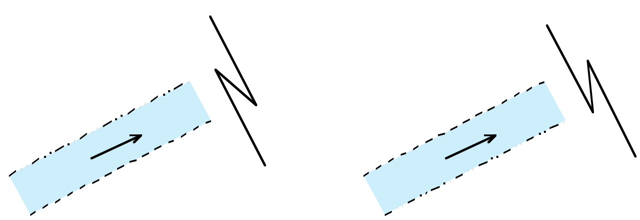
Complexity

- Satisfying M-V=±2 is “easy”
- Satisfying alternate angle sums is “easy”
- Satisfying layer order (M-V assignment) is “hard”…

- How hard?

Pleats as logical signals

- Two parallel pleats must be opposite parity
- For a specified direction, there are 2 allowed crease assignments



Valley on right = “true” Valley on left = “false”

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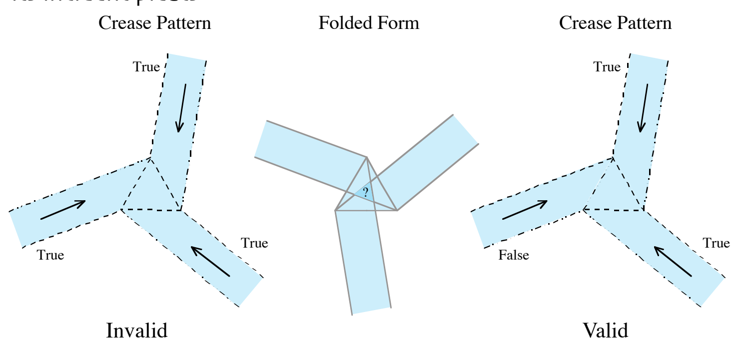
Not-All-Equal

- A particular crease pattern enforces the condition "Not-All-Equal" on its incident pleats



- It is possible to create multiple such conditions, thereby encoding NAE logic problems as crease assignment problems

Crease Assignment Complexity

- Marshall Bern and Barry Hayes showed in 1996 that any NAE-3-SAT problem can be encoded as a crease assignment problem
- NAE-3-SAT is NP-complete!
- Ergo, "Origami is hard!"
-

  But most problems of interest are polynomial (still hard, but solvable)

P.S.

- Even if you have the complete crease assignment, simply determining a valid layer ordering is still NP-complete!

Flat-Foldability

- A crease pattern is "flat foldable" iff it satisfies:
  - Maekawa Condition (M-V parity) at every interior vertex
  - Kawasaki Condition (Angles) at every interior vertex
  - Justin Conditions (Ordering) for all facets and creases

Within this description, there are many interesting and unsolved problems!

But is it useful, or just fun?

- The mathematical progression:
- Flat-foldability rules (math)…
- lead to crease pattern matching rules (application)…
- and thus, the generation of beauty (art)…
- and even practical functional objects ($$$)!

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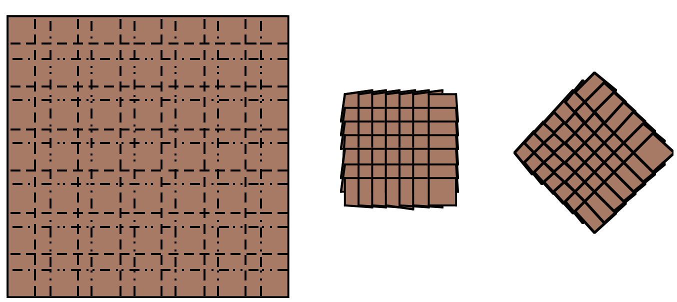
Textures

- Patterns of intersecting pleats can be integrated with other folds to create textures and visual interest



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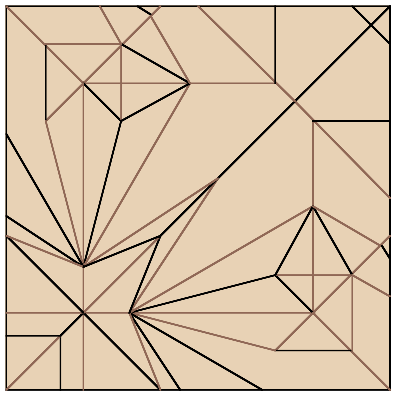
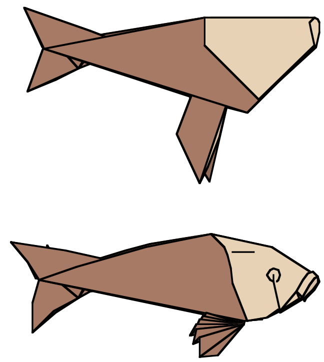
The recipient form





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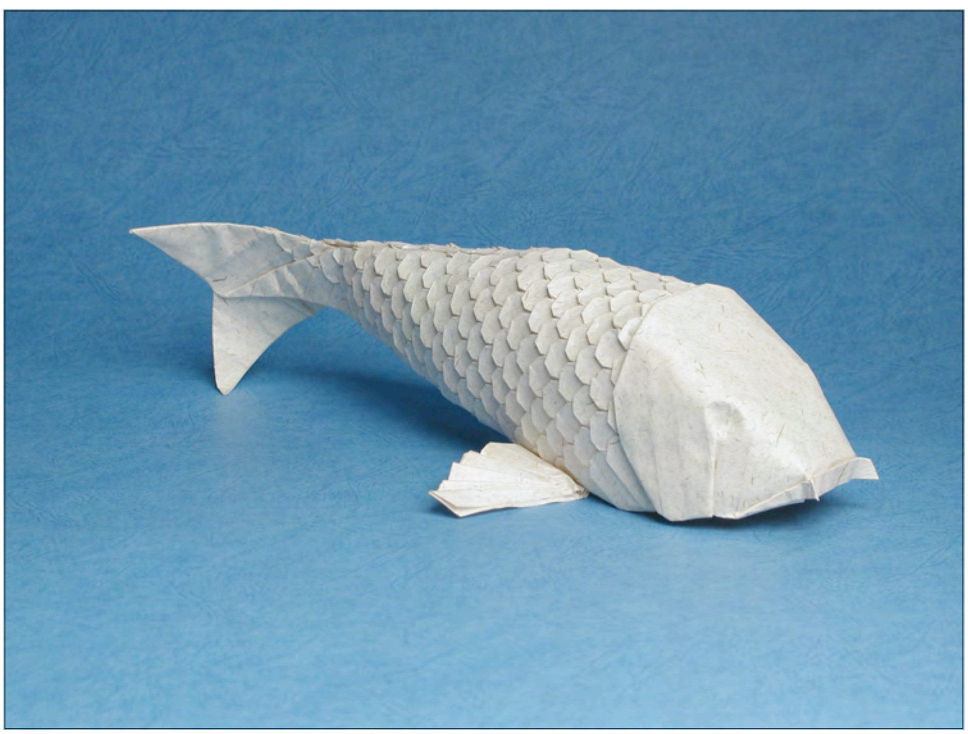
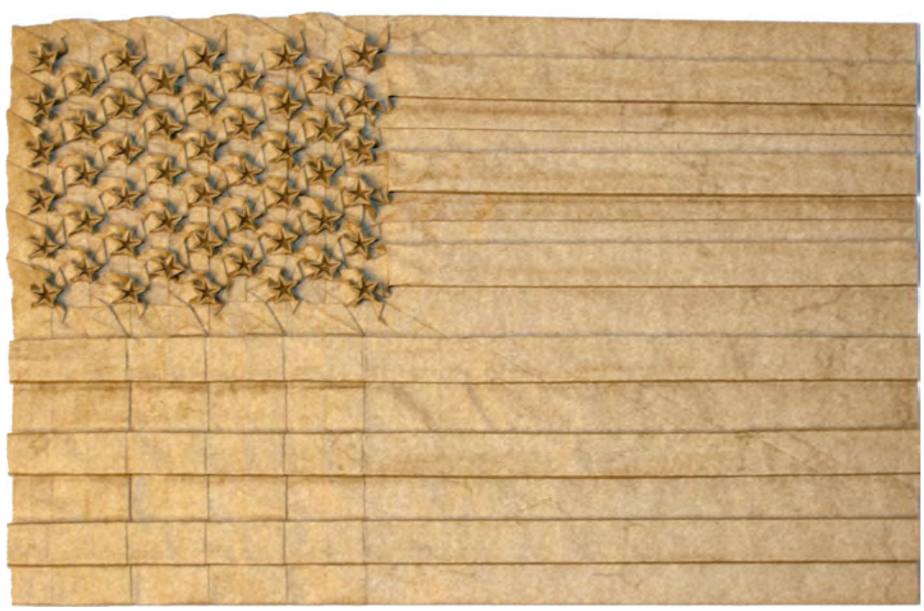


Western Pond Turtle



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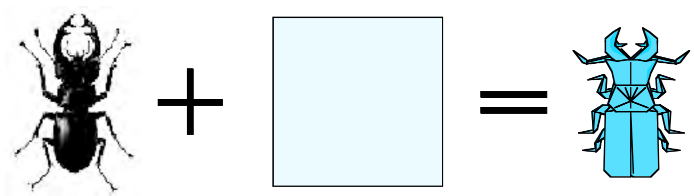
Rattlesnake

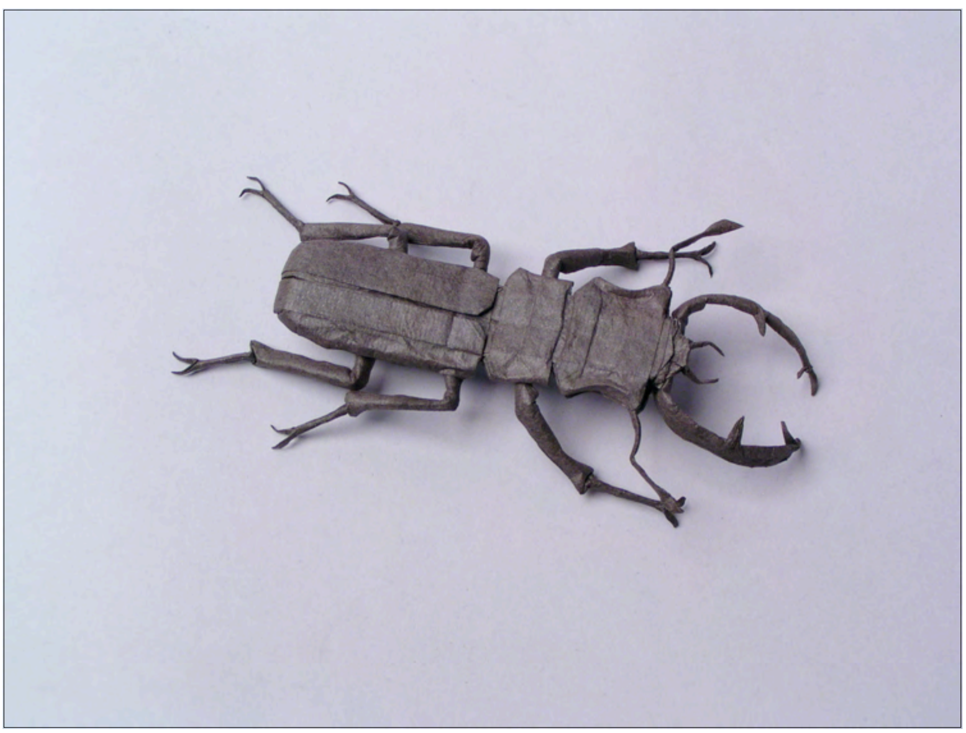
Flap Generation

- The most extensive and powerful origami tools deal with the generation of flaps in a desired configuration.
- Why is this useful?

Origami design

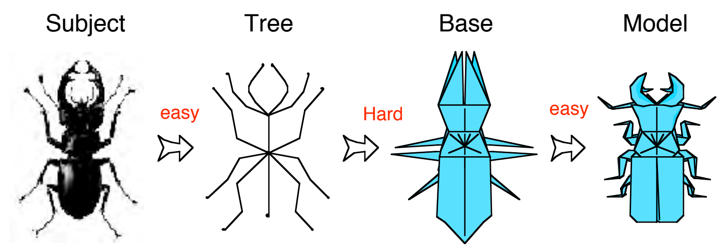
- The fundamental problem of origami design is: given a desired subject, how do you fold a square to produce a representation of the subject?





A four-step process

- The fundamental concept of design is the base
- The fundamental element of the base is the flap
  - From a base, it is relatively straightforward to shape the flaps into the appendages of the subject.
- The hard step is:
  - Given a tree (stick figure), how do you fold a Base with the same number, length, and distribution of flaps as the stick figure?



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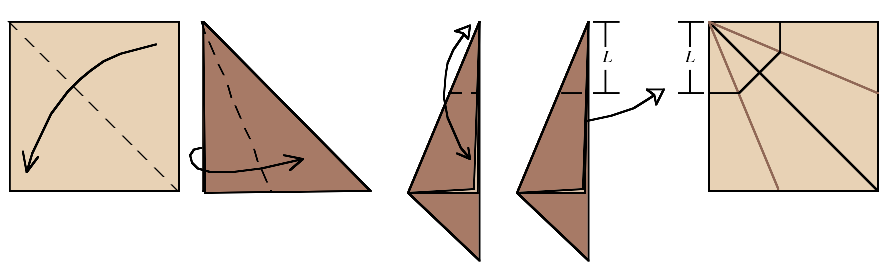
How to make a flap

- To make a single flap, we pick a corner and make it narrower.
- The boundary of the flap divides the crease pattern into:
  - Inside the flap
  - Everything else
- “Everything else” is available to make other flaps



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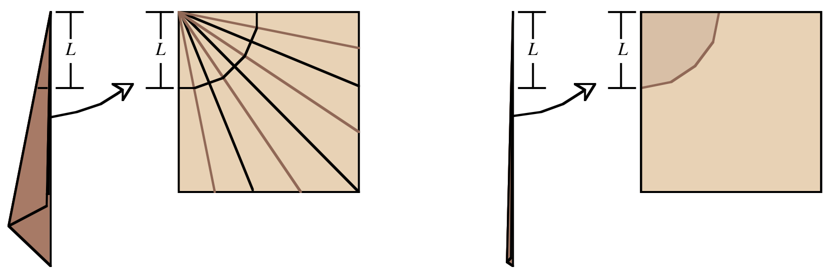
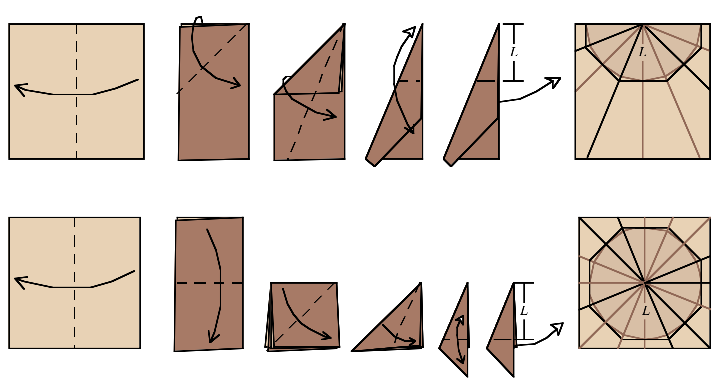
Limiting process

- What does the paper look like as we make a flap skinner and skinnier?
- A circle!



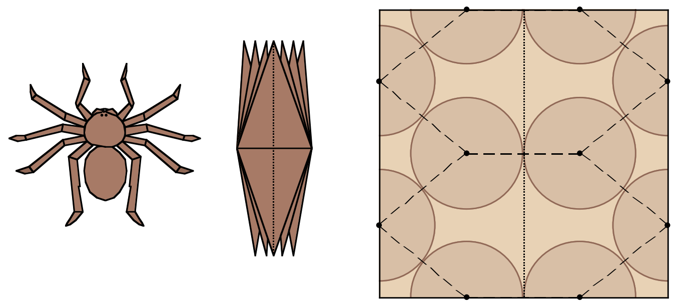
Other types of flap

- Flaps can come from edges…
- …and from the interior of the paper.



Circle Packing

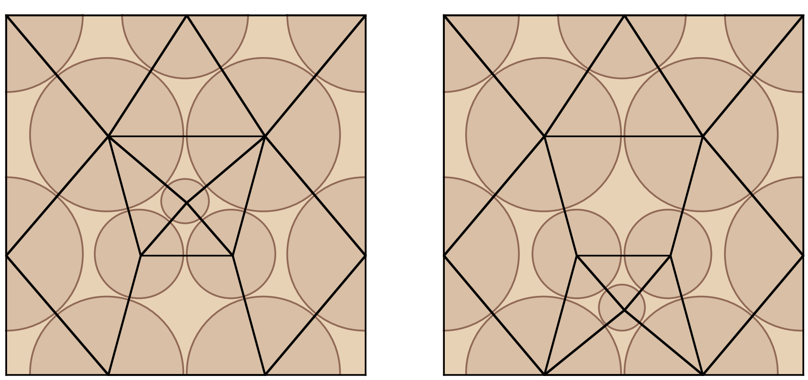
- In the early 1990s, several of us realized that we could design origami bases by representing all of the flaps of the base by circles overlaid on a square.



Subject Hypothetical Base Circle Packing

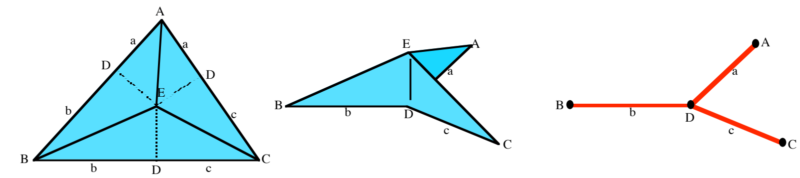
Creases

- The lines between the centers of touching circles are always creases.
- But there needs to be more. Fill in the polygons, but how?



Molecules

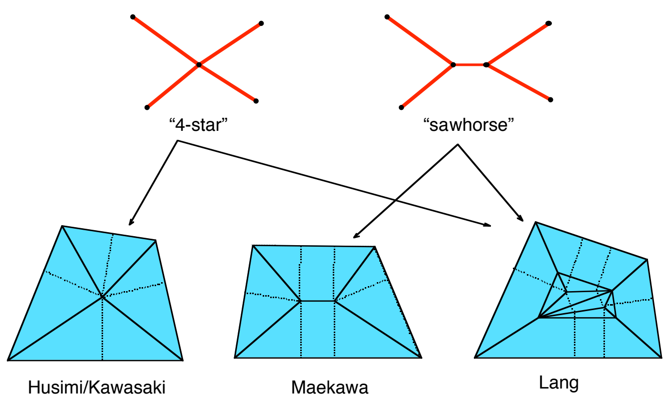
- Crease patterns that collapse a polygon so that its edges form a stick figure are called "bun-shi," or molecules (Meguro)
- Each polygon forms a piece of the overall stick figure (Divide and conquer).
- Different molecules are known from the origami literature.
- Triangles have only one possible molecule.



the "rabbit ear" molecule

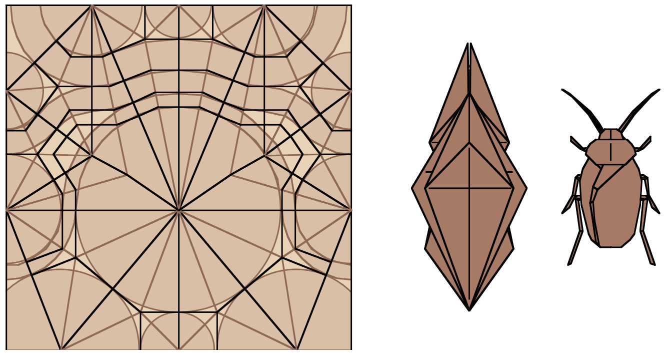
Quadrilateral molecules

- There are two possible trees and several different molecules for a quadrilateral.
- Beyond 4 sides, the possibilities grow rapidly.



Circles and Rivers

- Pack circles, which represent all the body parts.
- Fill in with molecular crease patterns.
- Fold!



Circle-River Design

- The combination of circle-river packing and molecules allows an origami composer to construct bases of great complexity using nothing more than a pencil and paper.
- But what if the composer had more…
- Like a computer?

Formal Statement of the Solution

- The search for the largest possible base from a given square becomes a well-posed nonconvex nonlinear constrained optimization:
  - Linear objective function
  - Linear and quadratic constraints
  - Nonconvex feasible region
- Solving this system of tens to hundreds of equations gives the same crease pattern as a circle-river packing:

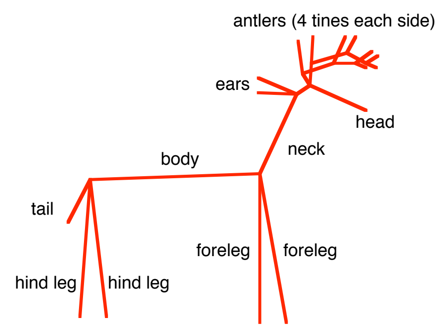
optimize m subject to:

$$m\, l_{ij} - \left[\left(u_{i,x} - u_{j,x}\right)^2 + \left(u_{i,y} - u_{j,y}\right)^2\right]^{1/2} \leq 0 \text{ for all } i, j$$

$$0 \leq u_{i,x} \leq 1,\; 0 \leq u_{i,y} \leq 1 \text{ for all } i$$

Computer-Aided Origami Design

- 16 circles (flaps)
- 9 rivers of assorted lengths
- 120 possible paths
- 184 inequality constraints
- Considerations of symmetry add another 16 more equalities
- 200 equations total!
- Child's play for computers.
- I have written a computer program, "TreeMaker," which performs the optimization and construction.



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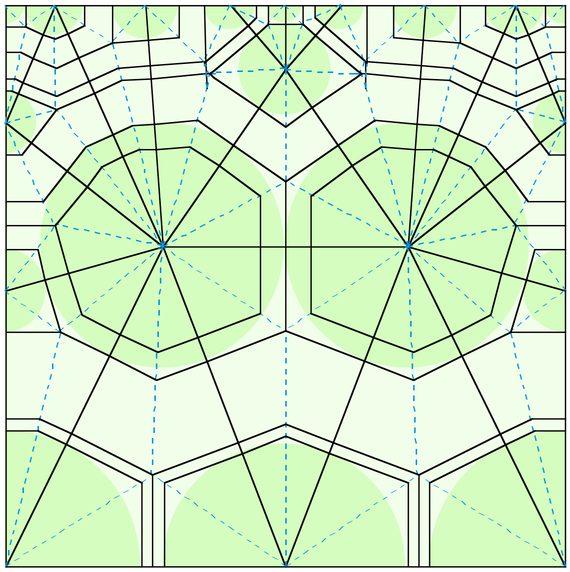
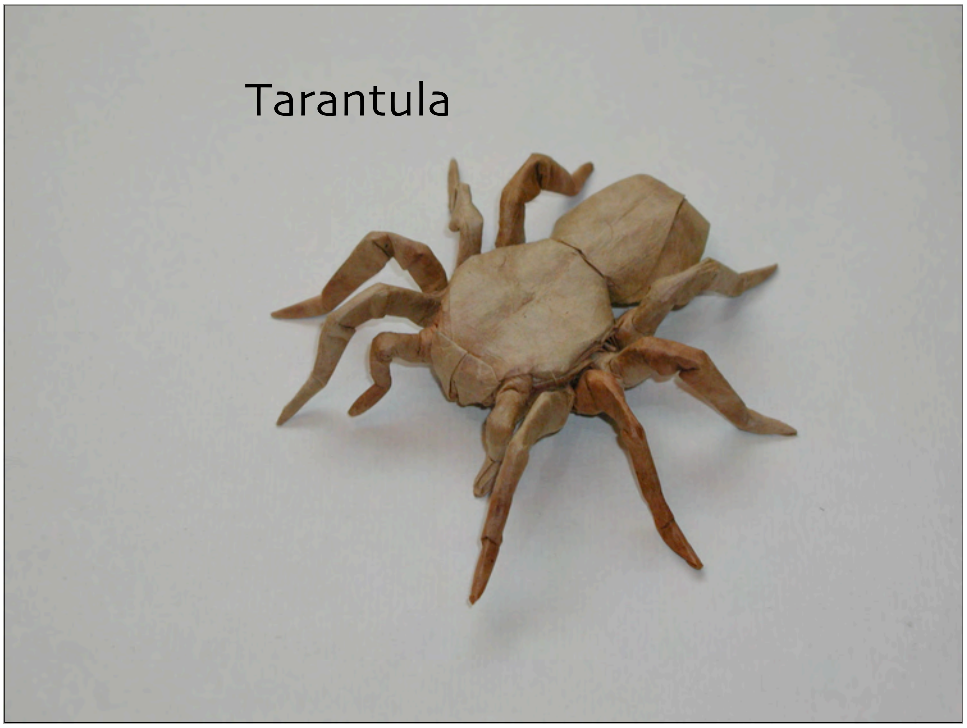
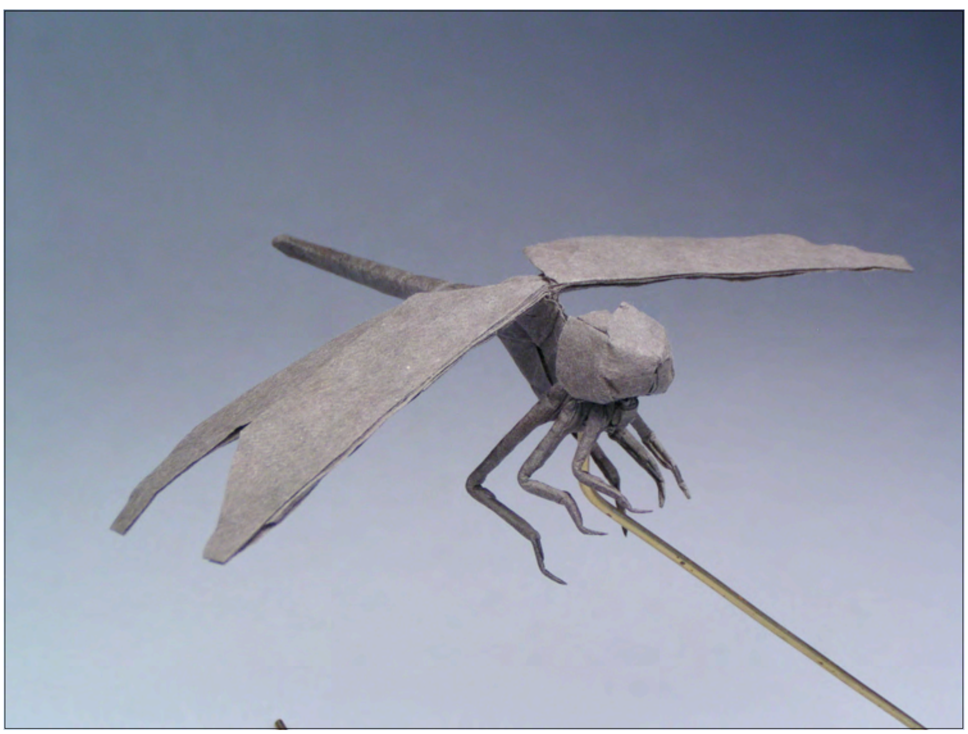
The crease pattern




Roosevelt Elk


Bull Moose

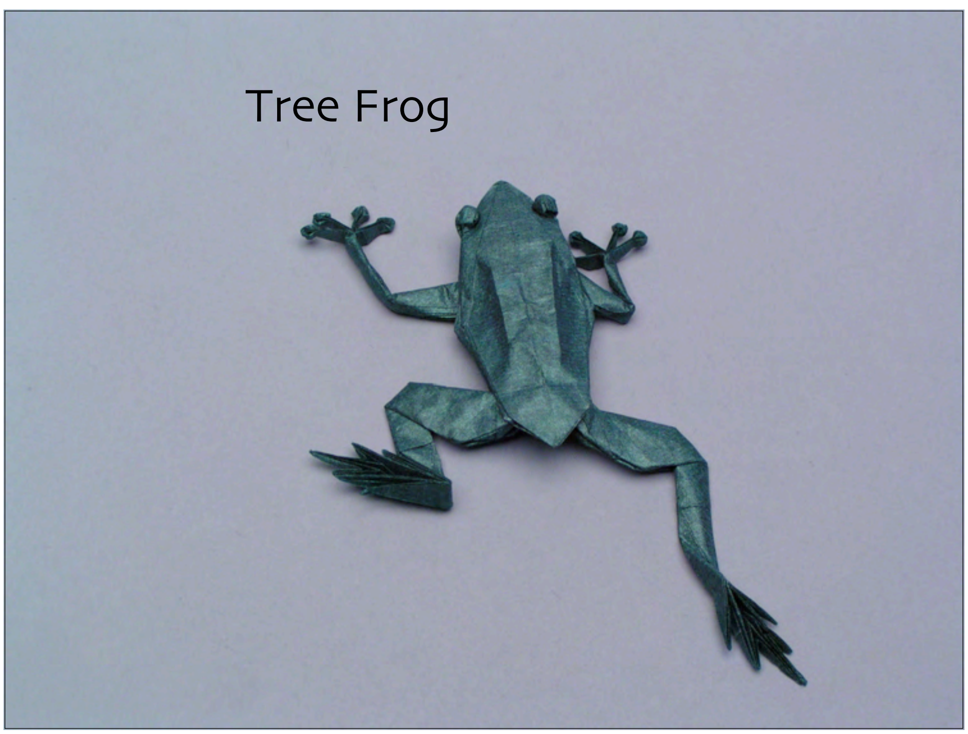

Tarantula

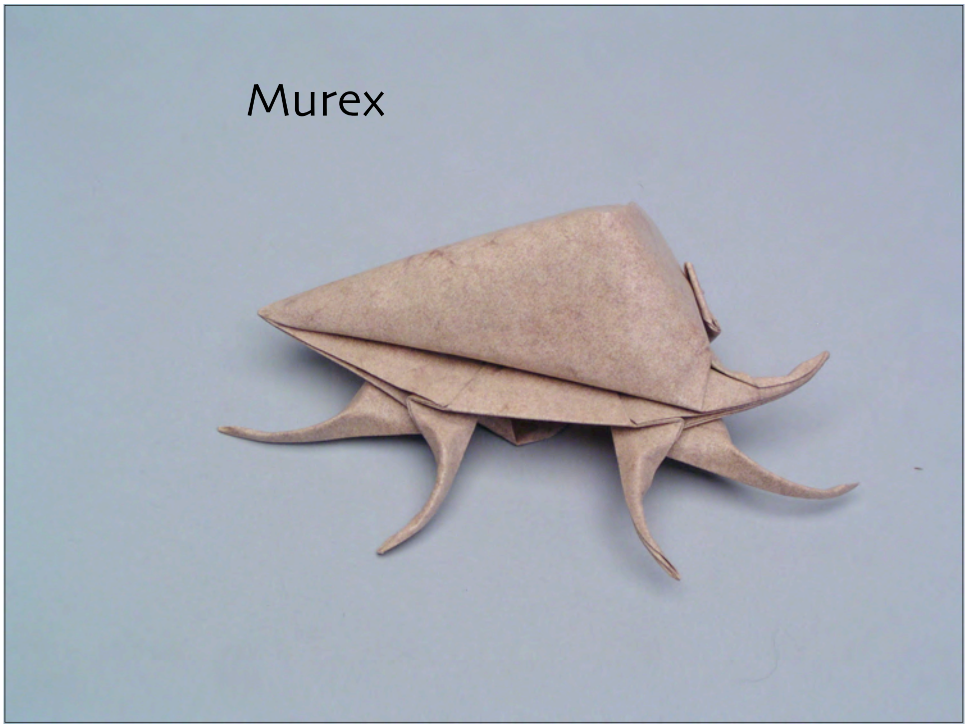


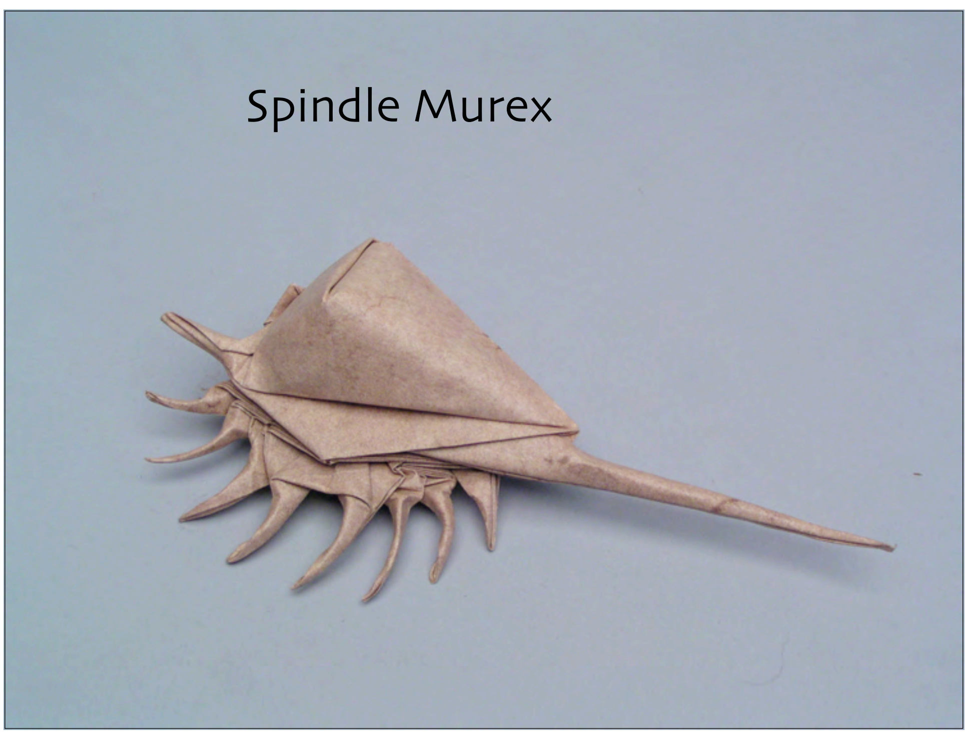

Praying Mantis

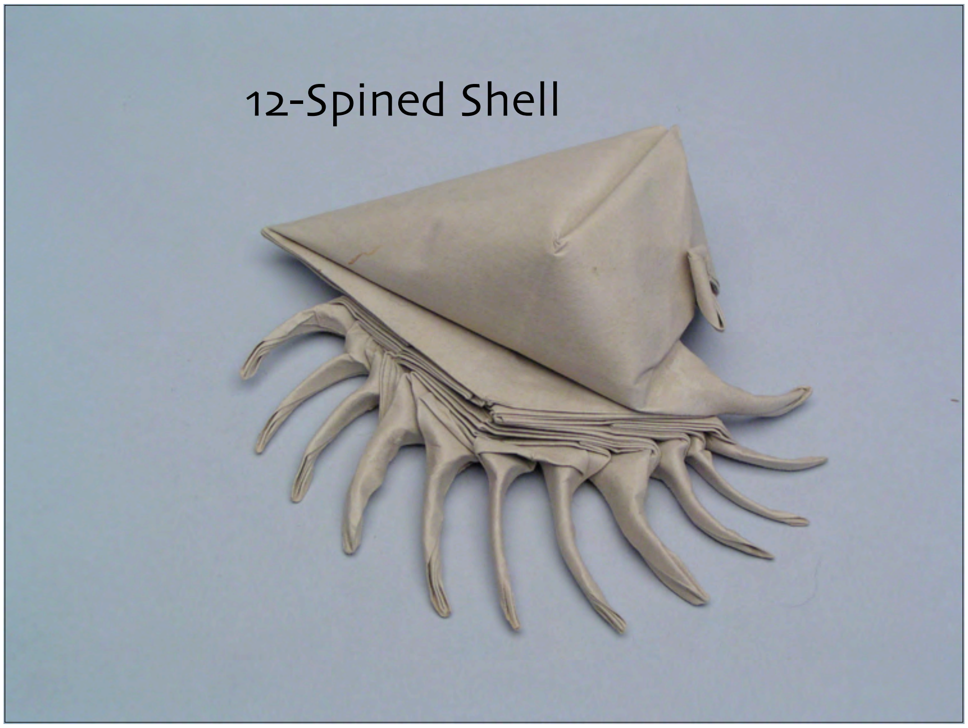

Two Praying
Mantises

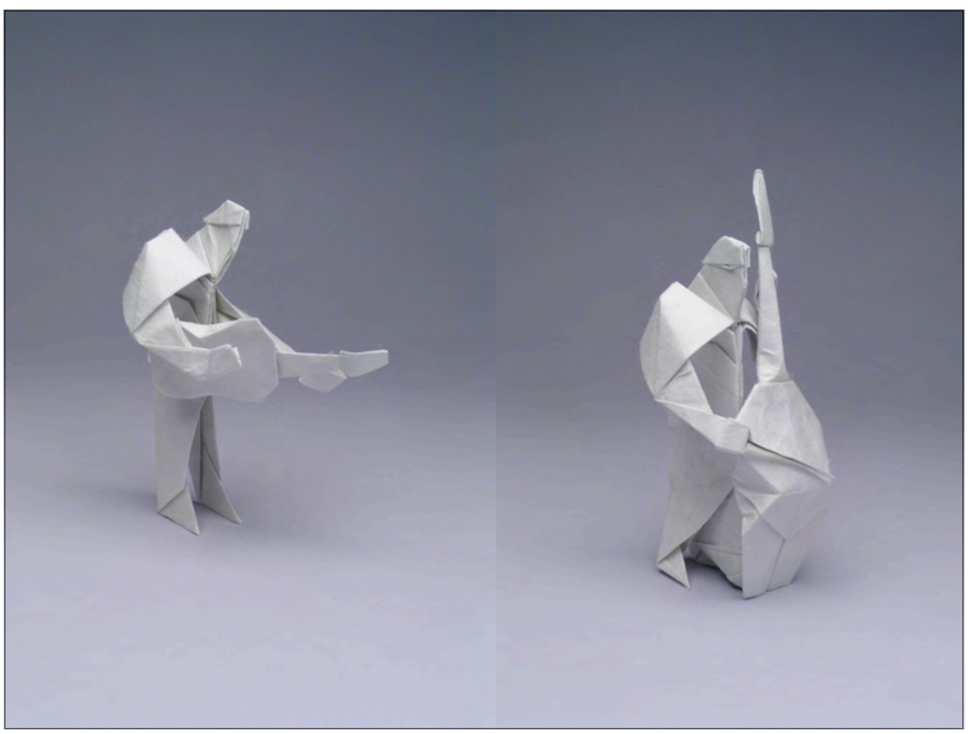

Grizzly Bear

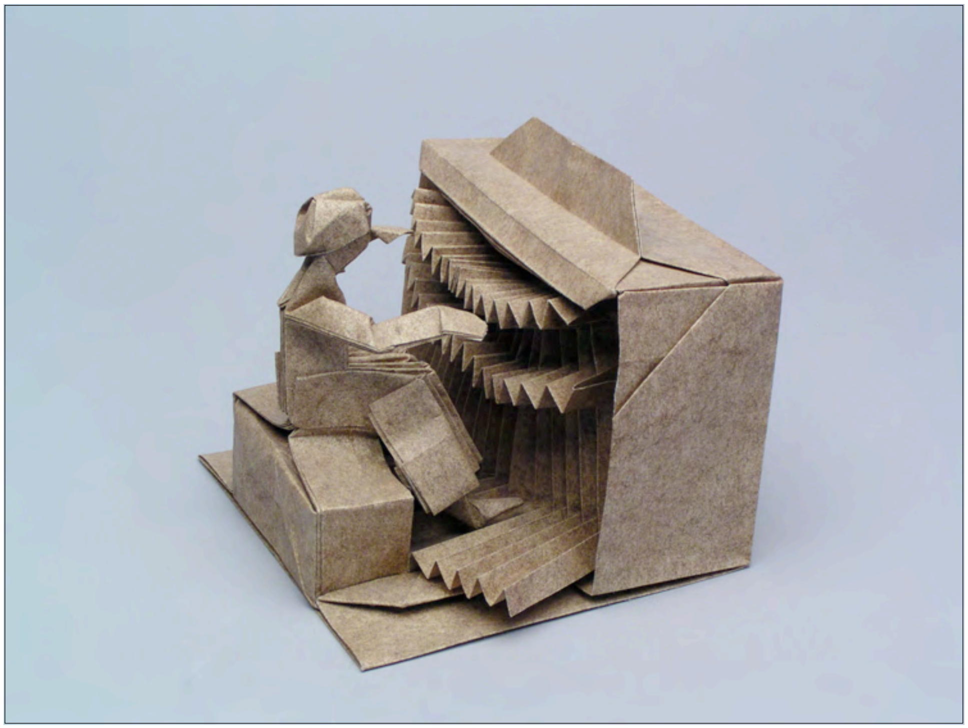

Tree Frog


Murex


Spindle Murex


12-Spined Shell



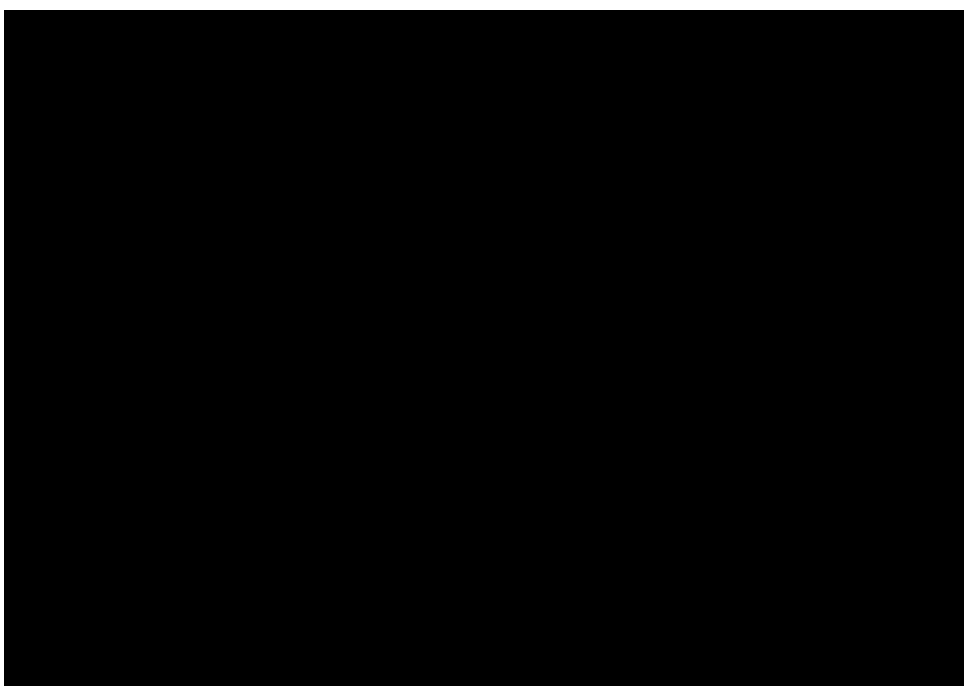


TreeMaker

- Algorithms are described in
  - R. J. Lang, "A Computational Algorithm for Origami Design," *12th ACM Symposium on Computational Geometry*, 1996
  - R. J. Lang, *Origami Design Secrets* (A K Peters, 2003)
- Macintosh/Linux/Windows binaries and source available (free!) from
  - http://www.langorigami.com/treemaker.htm

Origami on Demand

- Tools for origami design allow one to create an origami version of "almost anything"
- Recent years have seen origami commissioned for graphics, advertisements, commercials



Origami Software

- TreeMaker (Lang) -- shapes with appendages
- Origamizer (Tachi) -- arbitrary surfaces
- ReferenceFinder (Lang) -- finds folding sequences
- Tess (Bateman) -- constructs origami tessellations
- Rigid Simulator (Tachi) -- flexible surface linkages
- Oripa (Jun Mitani) -- crease pattern folder
- …and more!

Tachi's Teapot

The "Utah teapot"

Computed crease pattern

Geometric Origami

- Mathematical descriptions have permitted the construction of elaborate geometrical objects from single-sheet folding:
  - Flat Tessellations (Resch, Palmer, Bateman, Verrill)
  - 3-D faceted tessellations (Fujimoto, Huffman)
  - Curved surfaces (Huffman, Mosely)
  - …and more!

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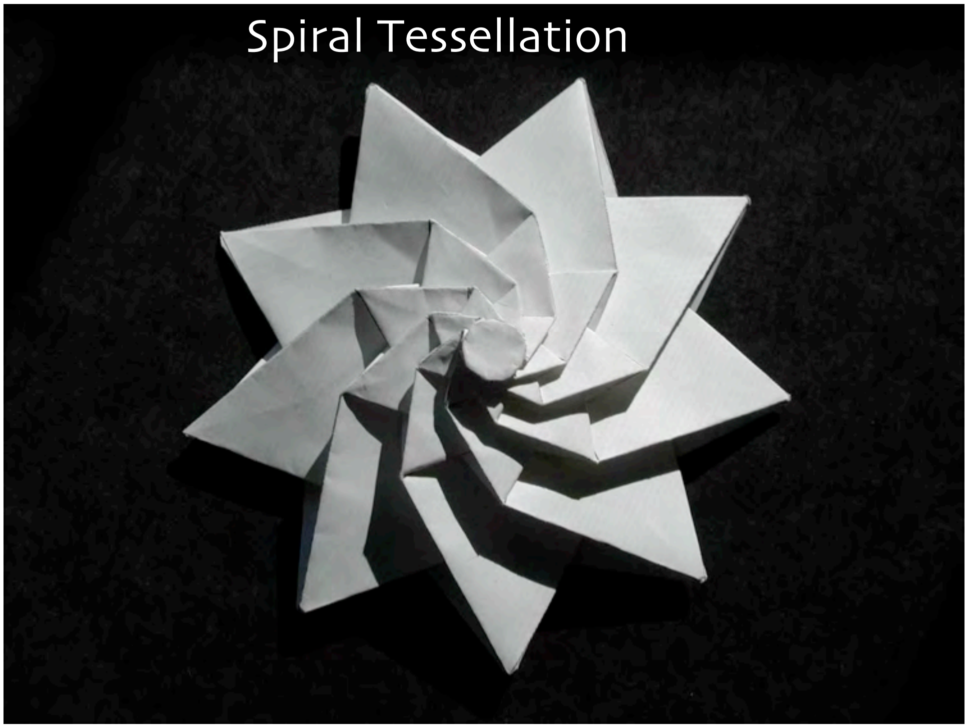
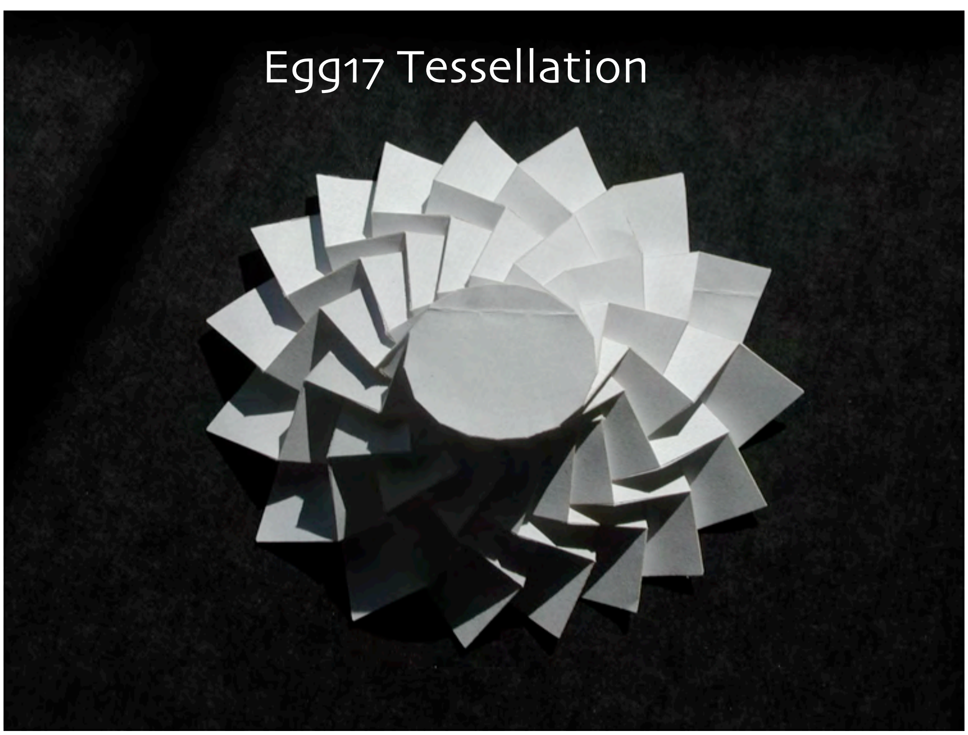
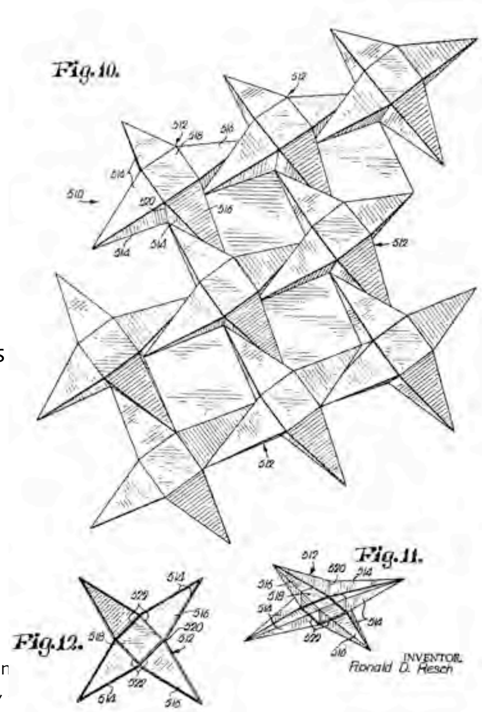

Spiral Tessellation

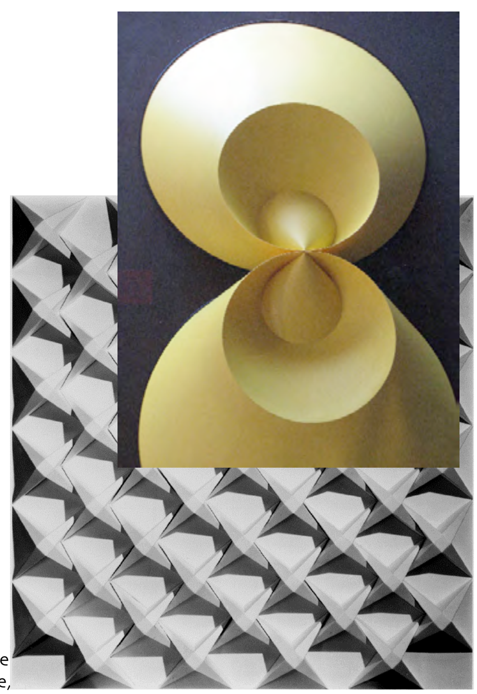

Egg17 Tessellation

Ron Resch

- Computer scientist and artist Ron Resch designed (and patented) 2- and 3-D tessellations back in the 1960s
- See US Patent 3,407,588.



Ron Resch

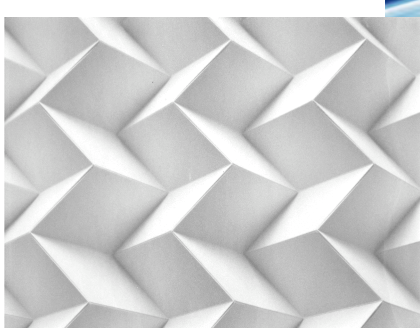


Applications in the Real World

- Mathematical origami has found many applications in solving real-world technological problems, in:
  - Space exploration (telescopes, solar arrays, deployable antennas)
  - Automotive (air bag design)
  - Medicine (sterile wrappings, implants)
  - Consumer electronics (fold-up devices)
  - …and more.
- Application in technology: origami rules don't matter
- …but no-cut-folding can be driven by technological reasons!

Muira-Ori, by Koryo Miura

- First "origami in space"
- Solar array, flew in 1995



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James Webb Space Telescope

- Multiply segmented mirror folds into thirds

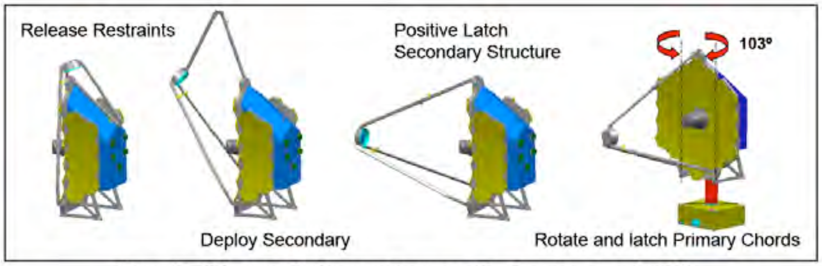
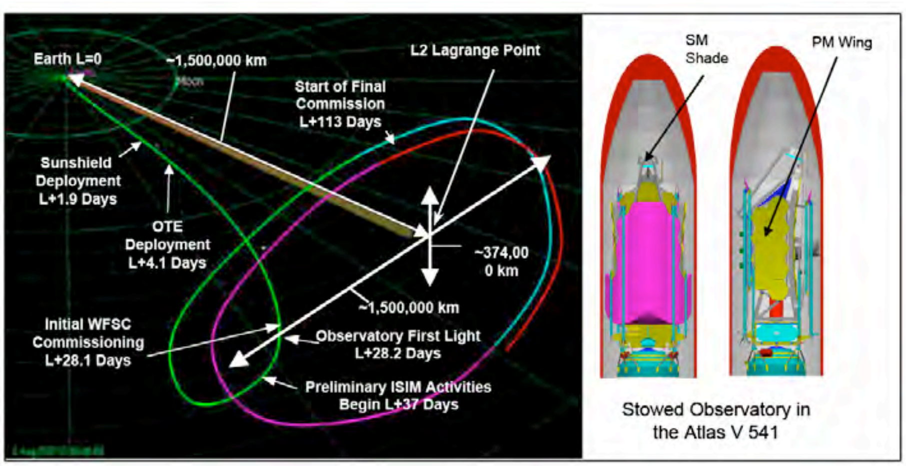


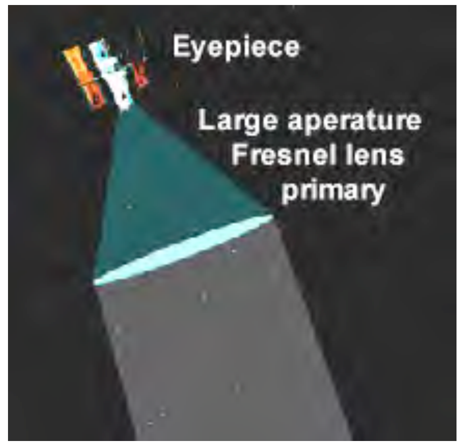
Figure 10. Telescope Deployment Sequence (Deployment steps 4 and 5)

JWST Stowage



The "Eyeglass" Telescope

- Under development at Lawrence Livermore National Laboratory
- 25,000 miles above the earth
- 100 meter diameter (a football field)
- Look up: see planets around distant stars
- Look down…



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The lens and the problem

- The 100-meter lens must fold up to 3 meters (shuttle bay)
- Lens must be made from ultra-thin sheets of glass with flexures along hinges
- What pattern to use?

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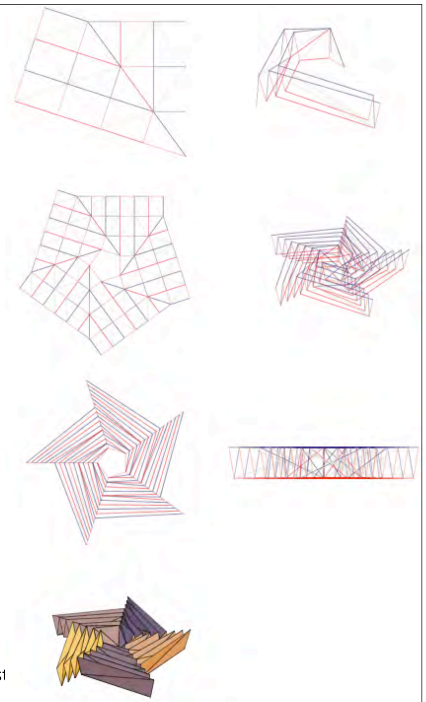
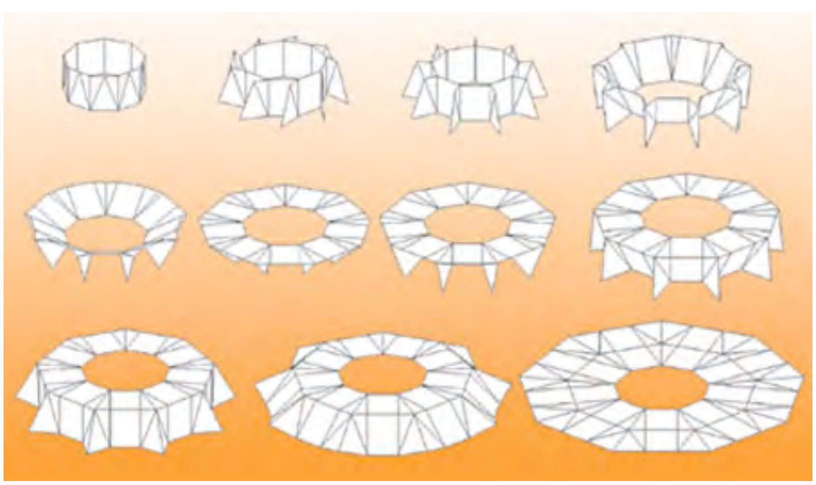
Analysis

- Analyzed several families of collapsing structures, including "flashers" and umbrella-liked patterns
- Initial modeling in *Mathematica*™ solving NLCO that enforce isometry between folded and unfolded state, followed by 3D modeling at LLNL

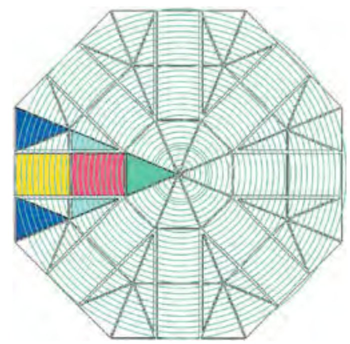


Umbrella



Manufacturability

- "Umbrella" was selected based on manufacturability issues
- Non-origami issues drive applications of origami



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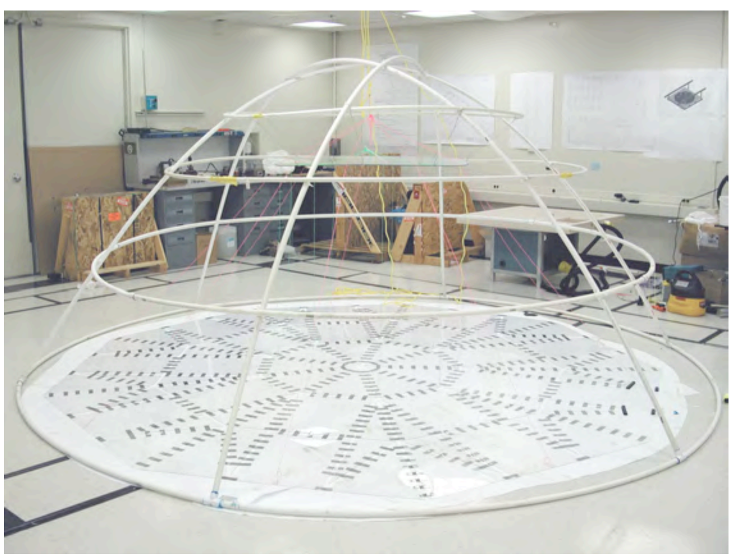
Foldable 3.7 meter Eyeglass



5-meter prototype

- The 5-meter prototype folds up to about 1.5 meter diameter.

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Solar Sail

- Japanese Aerospace Exploration Agency
- Mission flown in August 2004
- First deployment of a solar sail in space
- Pleated when furled, expands into sail

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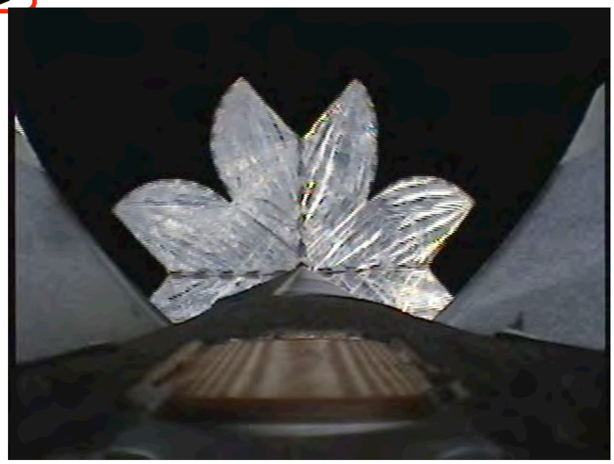
Solar Sail



http://www.isas.jaxa.jp/e/snews/2004/0809.shtml

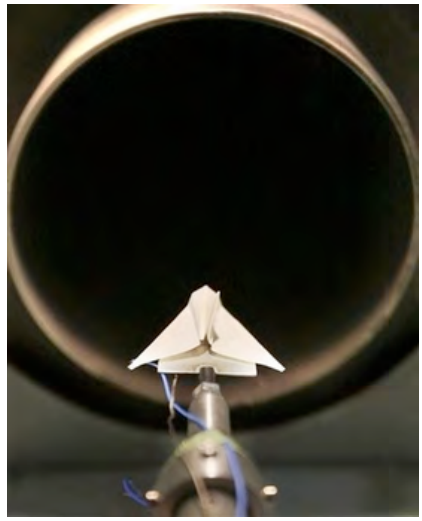
NASA Sail

- NASA, too, is developing unfolded and inflatable solar sails.

Video courtesy Dave Murphy, AEC-Able Engineering, developed under NASA contract NAS803043

Paper Airplanes

- JAXA approved “paper airplane” from space studies
- Prototype has survived Mach 7 and 446°F temperature!
- Tracking?



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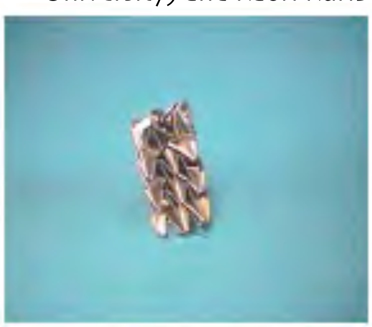
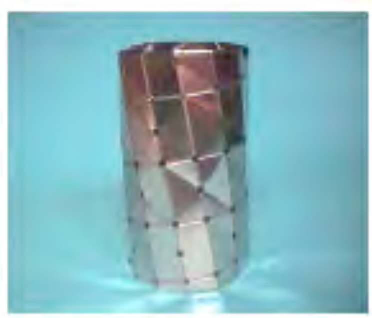
Stents

- Origami Stent graft developed by Zhong You (Oxford University) and Kaori Kuribayashi





An origami stent made from stain-less steel. Its diameter expends from 12 mm to 23 mm.

www.tulane.edu/~sbc2003/pdfdocs/0257.PDF

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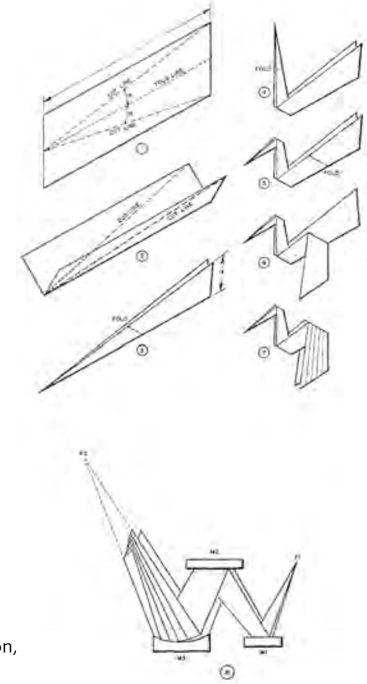
Optics

- “Optigami” -- simulation of optical systems using origami reverse folds
- --Jon Myer, Hughes Research Laboratories, *Applied Optics*, 1969



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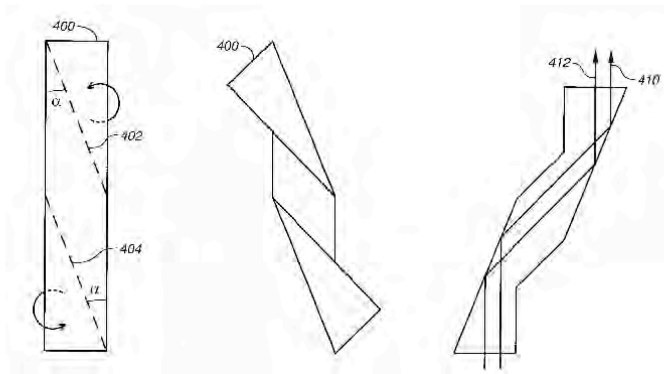
Lasers

- “Folded Cavity Laser” produces higher brightness than conventional broad-area semiconductor lasers



U.S. Patent 6,542,529 by Mats Hagberg and Robert J. Lang

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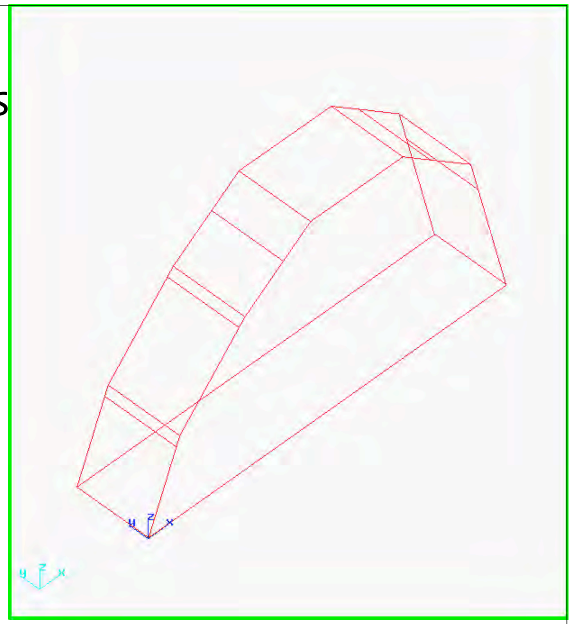
Airbags

- A mathematical algorithm developed for origami design turned out to be the proper algorithm for simulating the flat-folding of an airbag.



Animation courtesy EASi Enginering GmbH

Airbag Algorithm

- The airbag-flattening algorithm was derived directly from the universal molecule algorithm used in insect design.
- More complex airbag shapes (nonconvex) can be flattened using derivatives of Erik Demaine's fold-and-cut algorithm.
- No one foresaw these technological applications.
- (Not uncommon in mathematics!)

Resources

- Further information may be found at **http://www.langorigami.com**, or email me at robert@langorigami.com